Written Exam for the M.Sc. in Economics Autumn 2011-2012 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: 21 February 2012

3-hour open book exam.

All questions need to be answered.

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question 1:

Question 1.1: Consider the following model for log-returns, y_t

$$y_t = \sigma_t z_t \tag{1}$$

$$\sigma_t^2 = \omega + \begin{cases} y_{t-1}^2 & \text{if } s_t = 0 \\ \alpha y_{t-1}^2 & \text{if } s_t = 1 \end{cases} \tag{2}$$

The s_t is an unobserved state variable that can take values 0 or 1. Denote by p_t the probability that $s_t = 1$ conditional on the past returns, $p_t = P(s_t = 1 | y_{t-1})$.

We will assume that $p_t \to 1$ as $y_{t-1}^2 \to \infty$. That is, large lagged squared returns imply $s_t = 1$ with a high probability. As parameters we set $\theta = \{\omega, \alpha\}$, $\omega > 0$, $\alpha \geq 0$.

Show that y_t is weakly mixing with $Ey_t^2 < \infty$ if $\alpha < 1$.

Give an interpretation of this result and of the model.

Question 1.2: Corresponding to $\sigma_{0,t}^2 := \omega + y_{t-1}^2$, $\sigma_{1,t}^2 := \omega + \alpha y_{t-1}^2$, we can define the two Gaussian densities,

$$\phi_{0,t} = \frac{1}{\sqrt{2\pi\sigma_{0,t}^2}} \exp\left(-\frac{y_t^2}{2\sigma_{0,t}^2}\right), \quad \text{and} \quad \phi_{1,t} = \frac{1}{\sqrt{2\pi\sigma_{1,t}^2}} \exp\left(-\frac{y_t^2}{2\sigma_{1,t}^2}\right). \tag{3}$$

With the log-likelihood function, $L_T(\theta) = \sum_{t=1}^T l_t(\theta)$, state $l_t(\theta)$ in terms of $\phi_{0,t}, \phi_{1,t}$ and p_t.

Question 1.3: Find $s_\alpha := \partial L_T(\theta) / \partial \alpha$, that is the score in the direction of α, and show that, up to a constant,

$$s_\alpha = \sum_{t=1}^T p_t^* \left(\frac{y_t^2}{\sigma_{1,t}^2} - 1\right) \frac{y_t^2}{\sigma_{1,t}^2}. \tag{4}$$

In particular, provide an expression for p_t^* and explain what it is.

Discuss an algorithm as to how the MLE $\hat{\alpha}$ may be found.

Question 1.4:

Recall that p_t is a function of lagged returns y_{t-1}, and such that $p_t \to 1$ when $y_{t-1}^2 \to \infty$. Give an example of a non-constant probability function $p_t = p(y_{t-1})$ which satisfies this.

Explain how you would simulate y_t based on your (or any) choice of p_t.

With a simulated series of this form, y_t^{sim}, say, what would you expect to find empirically when fitting a GARCH(1,1) model to the y_t^{sim} data?

Question 2:

Question 2.1:
Consider the Realized volatility series v_t corresponding to $T = 796$ days in Figure 2.1 below together with the empirical ACF for x_t.

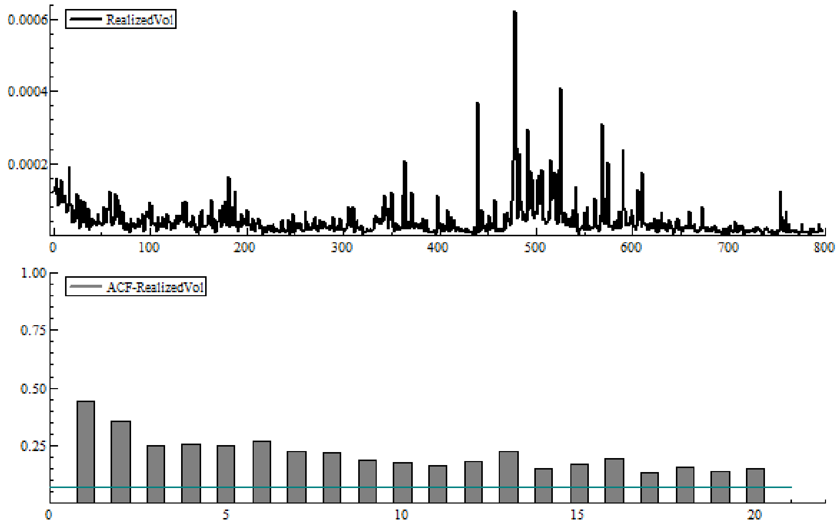

Figure 2.1

A preliminary analysis of v_t gives, that a model of the form,

$$v_t = \rho_1 v_{t-1} + \rho_{10} v_{t-10} + \rho_{32} v_{t-32} + \varepsilon_t, \tag{5}$$

with ε_t iidN$(0, \sigma^2)$, describes well the dynamics of v_t with the estimated coefficients $\hat{\rho}_1, \hat{\rho}_{10}$ and $\hat{\rho}_{32}$ all significant.

Discuss what type of memory, or autocorrelation structure, in v_t is being modelled and why given the graphs in Figure 2.1.

Suggest an alternative model with fewer parameters, and at the same time, more lags of v_t entering.

Question 2.2: Consider instead a different model for realized volatility v_t, as given by,

$$\Delta v_t = \pi v_{t-1} + \sigma_{s_t} z_t \tag{6}$$

where z_t is iidN$(0, 1)$, s_t is iid with $p(s_t = 1) = p_1 = 1 - p(s_t = 2)$, and σ_{s_t} switches between two values σ_1 and σ_2, with $\sigma_1 \neq \sigma_2$.

Define $\varepsilon_t = \sigma_{s_t} z_t$. State conditions under which ε_t is stationary.

Find $\sigma_\varepsilon^2 = E\varepsilon_t^2$ and $E\varepsilon_t\varepsilon_{t-1}$.

Consider the recursion, $v_t^* := \sum_{i=0}^{\infty} (1+\pi)^i \varepsilon_{t-i}$. Show that $E(v_t^*)^2 < \infty$ if $\pi \in]-2,0]$ and ε_t is stationary.

It can be shown that this implies that v_t^* is well-defined stationary and weakly mixing process. What is the relation with v_t?

Question 2.3: Rather than the simple iid switching assume let s_t be Markov switching according to the transition matrix, P,

$$P = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix}. \tag{7}$$

The parameters of the model are now given by $\theta = (p_{11}, p_{22}, \pi, \sigma_1^2, \sigma_2^2)$.

Explain how you would find the MLE $\hat{\theta}$.

Question 2.4:

Estimation of the Markov switching model for realized volatility v_t gave the following:

$\underset{\text{[std. error]}}{\hat{p}_{ij}}$:	$\hat{p}_{11} = \underset{[0.015]}{0.91}$	$\hat{p}_{21} = \underset{[0.044]}{0.25}$
$\underset{\text{[std. error]}}{\hat{\sigma}_{s_t}}$ $\times 10^5$:	$\hat{\sigma}_1 = \underset{[0.6]}{1.3}$	$\hat{\sigma}_2 = \underset{[2.7]}{8.7}$
$\hat{\pi}$:	-0.16	

Also LM-tests based on estimated standardized residuals, $\hat{z}_t$, gave:

LM no ARCH:	p-value: 0.001
Test of normality:	p-value: 0.02

Comment on the output.

Question 2.5:

Ignoring the Markov switching, the model is rewritten as,

$$\Delta v_t = \pi v_{t-1} + \epsilon_t, \tag{8}$$

with ϵ_t iidN$(0, \sigma_\epsilon^2)$. As usual in realized volatility modeling, v_t may be viewed as sampled in discrete time time t ($t = 1, 2, 3, ..., T = 800$) from a continuous time counterpart on the form:

$$dV_u = bV_u du + c dW_u, \tag{9}$$

where $u \in [0, T]$ and W_u is a Brownian motion.

Use Ito's formula to find the link between b and π.

Next compute the MLE $\hat{b}$.

For which values of b would you expect V_u to be stationary?

Explain how would you test if the continuous time process is a Brownian motion.